1. Consider a free spin 1/2 particle, i.e. $H_0 = \frac{p^2}{2m}$

a. What is the 2×2 matrix representation of the S_x^2 operator in the eigenbasis of S_z operator? (6 points)

b. If we add a term to the Hamiltonian as $H_{int} = cS_x^2$, what are corrections to the energy eigenvalues in this new system? (6 points)

c. Let us start with a state as $|s_z = 1/2\rangle$ with momentum p_0, how does such a state evolve with time after we add H_{int}? (8 points)

2. Consider an electron running in the Coulomb potential and suppose the electron (spin 1/2) has an orbital angular momentum as $l = 1$.

a. What are possible values of the total angular momentum? (4 points)

b. Label the spin operator of the electron as $\vec{S}$ and the electron's orbital angular momentum operator as $\vec{L}$. Let us introduce an interaction as $H_{int} = c\vec{S} \cdot \vec{L}$. What are the energy eigenstates for such a system? (8 points)

c. Label the z-component of the orbital angular momentum as $m\hbar$, and label $j(j+1)\hbar^2$ as the eigenvalue of the square of the total angular momentum operator, $\vec{J} = \vec{L} + \vec{S}$. If we measure the total angular momentum, j, on the state $|l = 1, m = 1; s = 1/2, s_z = -1/2\rangle$, what is the probability of finding $j = 1/2$? (8 points)

3. Consider an electron moving in an electromagnetic field, characterized by the gauge potential as ϕ and $\vec{A}$. Let us perform a gauge transformation as $\phi' = \phi$ and $\vec{A}' = \vec{A} + \nabla\Lambda(\vec{x})$.

a. How is the wavefunction of the electron after the gauge transformation related to the old wavefunction? (6 points)

b. How does the Heisenberg equation of motion for $\frac{dx_i}{dt}$, $\frac{dx_i}{dt} = \frac{[x_i, H]}{i\hbar} = \frac{p_i - eA_i/c}{m}$, change under the gauge transformation? (6 points)

c. Demonstrate that $\langle \frac{dx_i}{dt} \rangle$ is unchanged under such a gauge transformation. (8 points)