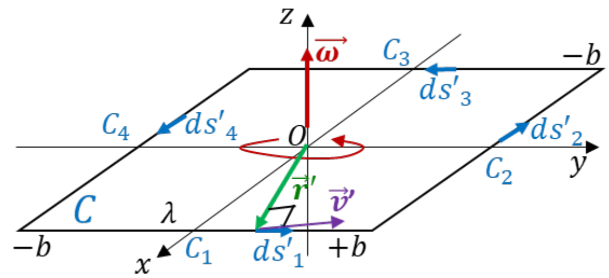
PHYS 7110 Fall 2022   Final/Comprehensive Exam   Dec. 15, 2022

Problem 1 [20 pts]



A thin square loop C of sides $2b$ as shown lies in the xy-plane. It is non-conducting and embedded with a uniform linear charge density λ. The loop is centered on, and is spinning about the z −axis at a constant angular speed ω in the counter-clock-wise sense as shown (i.e. $\vec{\omega} = \omega\hat{z}$). At the time shown, its sides are parallel to the x-axis and y-axis.

Find the magnetic (dipole) moment $\vec{m}$ of this spinning loop following the steps below.

Clearly the loop C, can be considered to be the sum of 4 simple paths C_1, C_2, C_3, C_4, that comprise the 4 sides of the square as shown. So that the dipole moment $\vec{m} = \vec{m}_1 + \vec{m}_2 + \vec{m}_3 + \vec{m}_4$ is the just sum of the four parts. There is clearly symmetry over the four segments.

We will start by considering $\vec{m}_1$ from the front segment C_1. Make all your calculations at the time shown.

*** Note that this problem does not reduce to a linear current circulating in the square loop.

(a) [3 pts] For the line element ds'_1 (note these are scalars in this problem) on C_1 located at y' to the right of the x-axis as shown, write down the position vector $\vec{r}'$ in terms of Cartesian coordinates (x', y', and/or z') plus given constants, AND in Cartesian components (i.e. as a linear combination of unit vectors $\hat{x}$, $\hat{y}$, and/or $\hat{z}$).

*** Note we are using primed coordinates as usual to indicate position within a charge/current distribution that generates an electric/magnetic field.

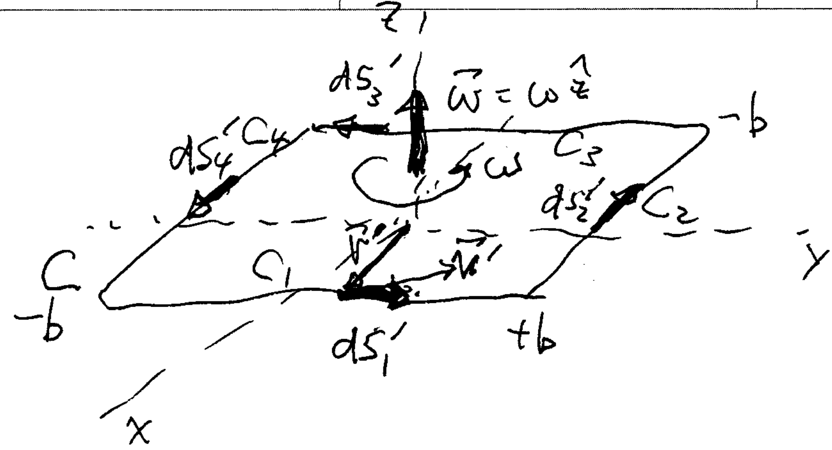
(b) [3 pts] Calculate the instantaneous velocity $\vec{v}'$ of the line element ds'_1, using the given $\vec{\omega}$ and the position vector $\vec{r}'$ in Cartesian coordinates and Cartesian components.

(c) [3 pts] Write the (3-scalar) line element ds'_1 in Cartesian coordinates (and/or their differentials), and hence write down the charge element dq' in terms of Cartesian coordinates (and/or their differentials), and given constants.

(d) [4 pts] Calculate the integrand for $\vec{m}_1$ that involves a further cross-product. Use Cartesian coordinates and components. Remember cross-products are vectors.

(e) [4 pts] Integrate over the straight line segment C_1 at the time shown (parallel to the y −axes) to find the magnetic moment $\vec{m}_1$. Hint: remember $\vec{m}_1$ is a vector.

(f) [3 pts] Use the symmetry of the system and you answer for $\vec{m}_1$ to write down solutions for $\vec{m}_2$, $\vec{m}_3$, and $\vec{m}_4$. Hence find the total dipole moment $\vec{m}$. Remember they are all VECTORS!!!



We note that $\vec{m} = \vec{m}_1 + \vec{m}_2 + \vec{m}_3 + \vec{m}_4$

$C \quad C_1 \quad C_2 \quad C_3 \quad C_4$

We will start by looking at C_1: runs ∥ to y-axis

(a) $\boxed{ds_1' = dy'}$ line element $\Rightarrow \boxed{dq' = \lambda\, ds_1' = \lambda\, dy'}$

y': from $-b$ to $+b$

$x' = b \quad z' = 0$

(b) location $\boxed{\vec{r}\,' = b\hat{x} + y'\hat{y}}$ $(z' = 0)$

(c) So for that line element

$$\vec{v}\,' = \vec{\omega} \times \vec{r}\,' = \omega\hat{z} \times (b\hat{x} + y'\hat{y})$$

$$\boxed{\vec{v}\,' = \omega(-y'\hat{x} + b\hat{y})}$$

(d) $$\vec{m}_1 = \frac{1}{2}\int_{C_1} \vec{r}\,' \times (dq'\vec{v}\,') = \frac{1}{2}\int_{C_1} dq'(\vec{r}\,'\times\vec{v}\,')$$

$$\vec{r}\,'\times\vec{v}\,' = (b\hat{x} + y'\hat{y}) \times \omega(-y'\hat{x} + b\hat{y})$$

$$\boxed{\vec{r}\,'\times\vec{v}\,' = \omega(b^2 + y'^2)\hat{z}}$$

(e) $$\vec{m}_1 = \frac{1}{2}\int_{C_1} dq'(\vec{r}\,'\times\vec{v}\,') = \frac{1}{2}\int_{-b}^{b} \overbrace{\lambda\, dy'}^{dq'} \cdot \omega(b^2 + y'^2)\hat{z}$$

... Cont'd

(e) cont'd

$$\vec{m}_1 = \frac{\lambda\omega}{2}\hat{z}\int_{-b}^{b}(b^2+y'^2)\,dy' = \frac{\lambda\omega}{2}\hat{z}\left[b^2y' + \frac{y'^3}{3}\right]_{-b}^{b}$$

$$= \hat{z}\frac{\lambda\omega}{2}\left[b^3 + \frac{b^3}{3} - (-b)^3 - \frac{(-b)^3}{3}\right]$$

$$= \hat{z}\frac{\lambda\omega}{2}\cdot\frac{8b^3}{3} = \boxed{\frac{4\lambda\omega b^3}{3}\hat{z}}$$

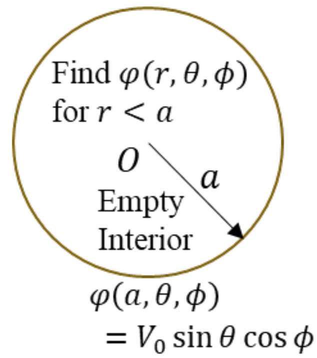
(f) Note $\vec{m}_1$ points only in the $\hat{z}$ direction

$\Rightarrow$ by symmetry

$$\vec{m}_2 = \vec{m}_3 = \vec{m}_4 = \vec{m}_1 = \frac{4\lambda\omega b^3}{3}\hat{z}$$

$$\Rightarrow \boxed{\vec{m} = \frac{16\lambda\omega b^3}{3}\hat{z}}$$

<u>Problem 2</u> [20 pts]



A hollow, non-conducting, thin sphere of radius a is centered on the origin. The surface (at radius a) is held at a non-isotropic potential given by

$$\varphi(a,\theta,\phi) = V_0 \sin\theta \cos\phi$$

Follow the steps below to find the potential in the empty interior (i.e. $r < a$) of the sphere. Note that θ is the *polar angle* measured from the $+z$-axis, while ϕ is the *azimuthal angle* measured in the xy-plane counterclockwise from the $+x$-axis.

(a) [5 pts.] Write down the most general solution $\varphi(r,\theta,\phi)$, to the Laplace Equation $\nabla^2\varphi = 0$, when solved by separation of variables in spherical coordinates r, θ, ϕ. This should be an infinite series summing over two indices, l and m. As we have done in class, use the coefficients A_{lm} for the non-negative (zero or positive) powers of r and B_{lm} for the negative powers of r.

(b) [5 pts] First apply the implicit boundary condition that the value of φ is finite at the origin. This should eliminate half of the coefficients (i.e. they are all zero for all values of l and m.). Indicate which coefficients vanish from this boundary condition and write the new, now restricted general solution for $r < a$.

(c) [10 pts] Now apply the stated boundary condition at $r = a$. Solve for the coefficients for $l = 0$ and $l = 1$, and all allowed values of m thereof.

Spherical Harmonics:

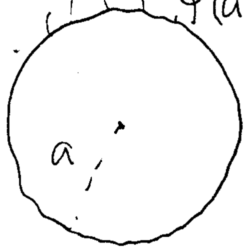
$$Y_0^0(\theta,\phi) = \frac{1}{2}\frac{1}{\sqrt{\pi}}$$

$$Y_1^{-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{-i\phi}$$

$$Y_1^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$$

$$Y_1^1(\theta,\phi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\, e^{i\phi}$$

$\varphi(a,\theta,\phi) = V_0 \sin\theta \cos\phi$



Find $\varphi(r,\theta,\phi)$ in the region $r<a$

Most General Solution:

(a) $\boxed{\varphi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^{l} + B_{lm} r^{-(l+1)} \right] Y_l^m(\theta,\phi)}$

(b) boundary condition @ $r=0$

$r \to 0, \quad r^{-(l+1)} \to \infty$

But $\varphi(0,\theta,\phi)$ should be finite:

$\Rightarrow \boxed{B_{lm} = 0}$ for all l, m.

$$\boxed{\varphi(r,\theta,\phi) = \sum_l \sum_m A_{lm} r^l Y_l^m(\theta,\phi)}$$

(c) $\varphi(a,\theta,\phi) = \sum_l \sum_m A_{lm} a^l Y_l^m(\theta,\phi) = V_0 \sin\theta \cos\phi$

Now we take $\int_{l'm'} d\Omega$

$$\int_0^{2\pi}\int_0^{\pi} Y_l^{*m}(\theta,\phi)\, \varphi(a,\theta,\phi) \sin\theta\, d\theta\, d\phi$$

$$= \sum_{l'} \sum_{m'} A_{l'm'} a^{l'} \int Y_l^{*m}(\theta,\phi) Y_{l'}^{m'}(\theta,\phi)\, d\Omega$$

$$= \sum_{l'} \sum_{m'} A_{l'm'} a^{l'} \delta_{ll'} \delta_{mm'} = a^l A_{lm}$$

$$\boxed{A_{lm} = \frac{1}{a^l} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, Y_l^{*m}(\theta,\phi)\, V_0 \sin\theta \cos\phi}$$

$$= \frac{V_0}{a^l} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, Y_l^{*m}(\theta,\phi) \sin^2\theta \cos\phi$$

(d) $l=0, m=0$:

$$Y^{*}(\theta,\phi) = \left[\sqrt{\frac{1}{4\pi}}\right]^{*}$$

$$A_{00} = \frac{V_0}{\cancel{a^0}_{1}} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{\frac{1}{4\pi}} \sin^2\theta \cos\phi$$

$$= V_0 \sqrt{\frac{1}{4\pi}} \cancel{\int_0^{2\pi} \cos\phi\, d\phi}^{\,0} \underbrace{\int_0^{\pi} \sin^2\theta\, d\theta}_{=\pi/2}$$

$$\boxed{A_{00} = 0 \,!}$$

$l=1, m=0$

$$Y_1^{*0}(\theta,\phi) = \left[\sqrt{\frac{2(1)+1}{4\pi}\frac{(1-0)!}{(1+0)!}}\, \underbrace{P_1^0(\cos\theta)}_{P_1(\cos\theta) = \cos\theta} \cancel{e^{i(0)\phi}}^{\,1}\right]^{*}$$

$$= \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$A_{10} = \frac{V_0}{a^1} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \cdot \sqrt{\frac{3}{4\pi}} \cos\theta \sin^2\theta \cos\phi$$

$$= \frac{V_0}{a}\sqrt{\frac{3}{4\pi}} \cancel{\int_0^{2\pi} d\phi \cos\phi}^{\,0} \int_0^{\pi} \cos\theta \sin^2\theta\, d\theta$$

$$\boxed{A_{10} = 0 \,!}$$

$l=1 \quad m=\pm 1$

$$Y_1^{*\pm1}(\theta,\phi) = \left[\sqrt{\frac{2(1)+1}{4\pi}\frac{(1\mp1)!}{(1\pm1)!}}\, P_1^{\pm1}(\cos\theta)\, e^{\pm i\phi}\right]^{*}$$

$m=1$

$$Y_1^{*1}(\theta,\phi) = \left[-\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}\right]^{*} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}$$

$$= -\sqrt{\frac{3}{8\pi}} \sin\theta (\cos\phi - i\sin\phi)$$

$$A_{11} = \frac{V_0}{a^1} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left[-\sqrt{\frac{3}{8\pi}} \sin^3\theta \cdot \cos\phi \cdot (\cos\phi - i\sin\phi)\right]$$

$$= \frac{-V_0}{a}\sqrt{\frac{3}{8\pi}} \int_0^{2\pi} (\cos^2\phi - i\sin\phi\cos\phi)\, d\phi \int_0^{\pi} \sin^3\theta\, d\theta$$

$$\int_0^{2\pi} \cos^2\phi \, d\phi = \pi, \quad \int_0^{2\pi} \sin\phi\cos\phi \, d\phi = \frac{1}{2}\sin^2\phi \Big|_0^{2\pi} = 0 \, !$$

$$\int_0^{\pi} \sin^3\theta \, d\theta = -\int_0^{\pi} (1-\cos^2\theta)\, d\cos\theta = -\left[\cos\theta - \frac{1}{3}\cos^3\theta\right]_0^{\pi}$$

$$= -\left[-1 + \frac{1}{3} - 1 + \frac{1}{3}\right] = -\left(-\frac{4}{3}\right) = \frac{4}{3}$$

$$\Rightarrow \boxed{A_{11} = \frac{-V_0}{a}\sqrt{\frac{3}{8\pi}} \cdot \pi \cdot \frac{4}{3} = -\sqrt{\frac{2\pi}{3}}\,\frac{V_0}{a}}$$

$m = -1$ $\quad Y_1^{*-1}(\theta,\phi) = \left[\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}\right]^* = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$

$$= \sqrt{\frac{3}{8\pi}}\sin\theta\,(\cos\phi + i\sin\phi)$$

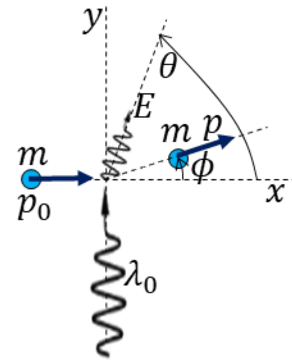
$$A_{1-1} = \frac{V_0}{a} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left[\sqrt{\frac{3}{8\pi}}\sin^3\theta \cdot \cos\phi\,(\cos\phi + i\sin\phi)\right]$$

$$= \frac{V_0}{a} \cdot \sqrt{\frac{3}{8\pi}} \int_0^{2\pi} (\cos^2\phi + i\sin\phi\cos\phi)\, d\phi \int_0^{\pi} \sin^3\theta\, d\theta$$

$$= \frac{V_0}{a}\sqrt{\frac{3}{8\pi}} \cdot \pi \cdot \frac{4}{3}$$

$$\boxed{A_{1-1} = \frac{V_0}{a}\sqrt{\frac{3}{8\pi}} \cdot \pi \cdot \frac{4}{3} = \sqrt{\frac{2\pi}{3}}\,\frac{V_0}{a}}$$

Problem 3 (20 pts)



The figure to the right shows the standard method of obtaining a tune-able gamma ray beam using the process of Inverse Compton Scattering. A beam of electron of original (magnitude of 3-) momentum p_0 and rest mass m travels in the positive $+x$ direction along the x-axis. A laser beam of original wavelength λ_0 travels in the $+y$ direction along the y-axis. The two beams collide elastically (i.e. the electron stays an electron) at the origin. The recoiling photon gets energy and momentum from the electron and now has new energy E, and travels at angle θ. The electron is scattered with (magnitude of 3-) momentum p, at angle ϕ, where both angles θ and ϕ are measured counter-clock-wise from the $+x$ axis in the xy-plane).

By selecting the photons (and placing a collimator) at a particular angle θ, you can choose the energy E of the new photon beam. This problem asks you to find E in terms of θ, the parameters λ_0, p_0, and the natural constants m, h and c, following the steps below.

(a) [4 pts] Write down the 4-momenta, P_e^μ and P_γ^μ of the electron and photon BEFORE the collision in terms of λ_0, p_0, m, h and c: i.e. in the form (replace the elements shown):

$$P_e^\mu = \begin{bmatrix} E_e/c \\ p_{ex} \\ p_{ey} \\ p_{ez} \end{bmatrix}, \qquad P_\gamma^\mu = \begin{bmatrix} E_\gamma/c \\ p_{\gamma x} \\ p_{\gamma y} \\ p_{\gamma z} \end{bmatrix}$$

(b) [4 pts] Write down the 4-momenta, P'^μ_e and P'^μ_γ of the electron and photon AFTER the collision in terms of E, p, m, h, c, θ and ϕ: i.e. in the form (replace the elements shown):

$$P'^\mu_e = \begin{bmatrix} E'_e/c \\ p'_{ex} \\ p'_{ey} \\ p'_{ez} \end{bmatrix}, \qquad P'^\mu_\gamma = \begin{bmatrix} E'_\gamma/c \\ p'_{\gamma x} \\ p'_{\gamma y} \\ p'_{\gamma z} \end{bmatrix}$$

*** Note in this problem, the "prime" (i.e. apostrophe) indicates quantities AFTER the collision, NOT those in a moving frame S'.

(c) [4 pts] Write down three independent equations involving E, p, θ and ϕ on the left hand side (LHS), and involving λ_0, p_0 on the right hand side (RHS), of each equation.

(d) [4 pts] Use two of the equations to eliminate ϕ and solve for p^2 (magnitude squared of the three – vector momentum) in terms of θ, E, p_0, m, h and c, then substitute your expression for p^2 into (and eliminate p, completely, from) the remaining equation.

(e) [4 pts] Algebraically solve for E in the remaining equation in terms of λ_0, p_0, m, h, c, and θ.

1(a) Incident electrons of momentum $p_e = p_0$

$$E_e/c = \sqrt{p_e^2 + m^2c^2} = \sqrt{p_0^2 + m^2c^2}$$

$$\boxed{P_e^{\mu} = \begin{bmatrix} \sqrt{p_0^2+m^2c^2} \\ p_0 \\ 0 \\ 0 \end{bmatrix}}$$

Original photon: λ_0 : $E_\gamma = \frac{hc}{\lambda_0}$, $p_\gamma = \frac{h}{\lambda_0}$

$$\boxed{P_\gamma^{\mu} = \begin{bmatrix} h/\lambda_0 \\ 0 \\ h/\lambda_0 \\ 0 \end{bmatrix}}$$

(b) e^- with new (3-vector magnitude) momentum $p_e' = p$

$$E_e'/c = \sqrt{p_e'^2 + m^2c^2} = \sqrt{p^2 + m^2c^2}$$

$$\boxed{P_e'^{\mu} = \begin{bmatrix} \sqrt{p^2+m^2c^2} \\ p\cos\phi \\ p\sin\phi \\ 0 \end{bmatrix}}$$

new photon with $E_\gamma' = E$, $\Rightarrow p_\gamma' = E/c$

$$\boxed{P_\gamma'^{\mu} = \begin{bmatrix} E/c \\ E/c \cos\theta \\ E/c \sin\theta \\ 0 \end{bmatrix}}$$

1 (c) 4-vector (total momentum) is conserved

$$\Rightarrow P_e^{\mu} + P_\gamma^{\mu} = P_e'^{\mu} + P_\gamma'^{\mu}$$

$$\begin{array}{ccc} \text{LHS} & & \text{RHS} \\ \begin{bmatrix} \sqrt{p^2+m^2c^2} + E/c \\ p\cos\phi + \frac{E}{c}\cos\theta \\ p\sin\phi + \frac{E}{c}\sin\theta \\ 0 \end{bmatrix} & = & \begin{bmatrix} \sqrt{P_0^2+m^2c^2} + h/\lambda_0 \\ P_0 \\ h/\lambda_0 \\ 0 \end{bmatrix} \end{array}$$

taking the x-row:

$$\boxed{p\cos\phi + \frac{E}{c}\cos\theta = P_0} \quad \cdots (1)$$

$$\boxed{p\sin\phi + \frac{E}{c}\sin\theta = h/\lambda_0} \quad \cdots (2)$$

$$\boxed{\sqrt{p^2+m^2c^2} + E/c = \sqrt{P_0^2+m^2c^2} + h/\lambda_0} \quad \cdots (3)$$

(d) $(1)^2 + (2)^2$

$$\Rightarrow p^2\cos^2\phi = \left(P_0 - \tfrac{E}{c}\cos\theta\right)^2 = P_0^2 - \frac{2P_0E}{c}\cos\theta + \frac{E^2}{c^2}\cos^2\theta$$

$$+)\ p^2\sin^2\phi = \left(h/\lambda_0 - \tfrac{E}{c}\sin\theta\right) = \frac{h^2}{\lambda_0^2} - \frac{2hE}{c\lambda_0}\sin\theta + \frac{E^2}{c^2}\sin^2\theta$$

$$p^2 = P_0^2 + \frac{h^2}{\lambda_0^2} - \frac{2P_0}{c} \quad \frac{2hE}{c\lambda_0}\sin\theta + \frac{E^2}{c^2}$$

sub into (3):

$$\Rightarrow \sqrt{p^2+m^2c^2} = \sqrt{P_0^2+m^2c^2} + \frac{h}{\lambda_0} - E/c$$

... cont'd

2(d)

$$\boxed{\Rightarrow \sqrt{P_0^2 + \frac{h^2}{\lambda_0^2} - \frac{2P_0E}{c}\cos\theta - \frac{2hE}{c\lambda_0}\sin\theta + \frac{E^2}{c^2} + m^2c^2} = \sqrt{P_0^2+m^2c^2} + \left(\frac{h}{\lambda_0} - \frac{E}{c}\right) \quad \cdots (4)}$$

2(e) squaring both sides of (4):

$$\cancel{P_0^2} + \cancel{\frac{h^2}{\lambda_0^2}} - \frac{2P_0E}{c}\cos\theta - \frac{2hE}{c\lambda_0}\sin\theta + \cancel{\frac{E^2}{c^2}} + \cancel{m^2c^2}$$

$$= \cancel{P_0^2} + \cancel{m^2c^2} + \left[2\sqrt{P_0^2+m^2c^2} \times \left(\frac{h}{\lambda_0} - \frac{E}{c}\right)\right] + \cancel{\frac{h^2}{\lambda_0^2}} - \frac{2hE}{c\lambda_0} + \cancel{\frac{E^2}{c^2}} \quad \cdots (5)$$

$$\Rightarrow \cancel{2}\sqrt{P_0^2+m^2c^2}\,\frac{h}{\lambda_0} - \frac{\cancel{2}\sqrt{P_0^2+m^2c^2}\,E}{c} - \frac{\cancel{2}hE}{c\lambda_0}$$ ← from RHS (5)

$$= \frac{-\cancel{2}P_0E}{c}\cos\theta - \frac{\cancel{2}hE}{c\lambda_0}\sin\theta$$ ← from LHS (5)

$$\Rightarrow \frac{hE}{c\lambda_0}(1-\sin\theta) + \frac{\sqrt{P_0^2+m^2c^2}}{c}E - \frac{P}{c}\cos\theta\, E = \sqrt{P_0^2+m^2c^2}\,\frac{h}{\lambda_0}$$

$$\boxed{E = \frac{\frac{h}{\lambda_0}\sqrt{P_0^2+m^2c^2}}{\frac{\sqrt{P_0^2+m^2c^2}}{c} + \frac{h}{c\lambda_0}(1-\sin\theta) - \frac{P_0}{c}\cos\theta} = \frac{ch\sqrt{P_0^2+m^2c^2}}{\lambda_0\sqrt{P_0^2+m^2c^2} + h - h\sin\theta - P_0\lambda_0\cos\theta}}$$

*** 9 blank pages will be provided to each student and they can insert additional blank pages (also provided) if needed.