Graduate Comprehensive Exam 2019 – Quantum Mechanics

Problem 1

A magnetic dipole with magnetic dipole moment operator $\boldsymbol{\mu} = \gamma\,\mathbf{S}$, where γ is a constant and $\mathbf{S}$ is the spin vector, precesses in a uniform magnetic field of magnitude B and direction $+z$. The Hamiltonian for this problem is

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}.$$

1. (5 points) Show that the Heisenberg equations of motion for the time-dependent components $\mu_x(t)$, $\mu_y(t)$, and $\mu_z(t)$ of the magnetic moment operator are

$$\frac{d\mu_x}{dt} = \omega\mu_y, \qquad \frac{d\mu_y}{dt} = -\omega\mu_x, \qquad \frac{d\mu_z}{dt} = 0,$$

   where $\omega = \gamma B$.

2. (5 points) Solve the Heisenberg equations of motion and express $\mu_x(t)$, $\mu_y(t)$, and $\mu_z(t)$ in terms of their initial values $\mu_x(0)$, $\mu_y(0)$, and $\mu_z(0)$.

3. (5 points) Show that the Ehrenfest equations for the expectation values $\langle\mu_x(t)\rangle$, $\langle\mu_y(t)\rangle$, and $\langle\mu_z(t)\rangle$ can be put into the form

$$\frac{d\langle\boldsymbol{\mu}\rangle}{dt} = -\boldsymbol{\omega} \times \langle\boldsymbol{\mu}\rangle, \tag{1}$$

   where $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$.

Problem 2

A system of two spin-1/2 particles is described by the Hamiltonian

$$H = A\big(\mathbf{S}_1 \cdot \mathbf{S}_2 - 3S_{1z}S_{2z}\big),$$

where A is a constant, $\mathbf{S}_1$ and $\mathbf{S}_2$ are the spin vectors of the particles, and S_{1z} and S_{2z} are their spin projections along the z axis.

1. (5 points) Express H in terms of $\mathbf{S}^2$ and S_z, where $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ is the total spin vector and S_z is its z component.

2. (5 points) Show that H commutes with $\mathbf{S}^2$ and S_z.

3. (5 points) Find the energy levels of the Hamiltonian H and their degeneracies.

Problem 3

Consider a rigid rotator free to rotate about the z axis. Its Hamiltonian is $H = L_z^2/(2I)$, where I is the rotator's moment of inertia and L_z is the z component of its orbital angular momentum.

1. (5 points) Find the energy eigenvalues and the normalized energy eigenfunctions (as functions of the azimuthal angle φ in the xy plane).

2. (5 points) Find the probability of each energy eigenvalue when the wave function is $\psi(\varphi) = A\cos^2\varphi$, where A is a constant.

3. (5 points) Find the wave function $\psi(\varphi, t)$ at time t if the initial wave function at $t = 0$ is the wave function $\psi(\varphi) = A\cos^2\varphi$ given in part 2.

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Problem 1

Consider a quantum simple harmonic oscillator of mass m and angular frequency ω. At time $t = 0$, the oscillator is in the state

$$|\psi\rangle = \exp\left(-\frac{ip\ell}{\hbar}\right)|0\rangle,$$

where p is the momentum operator, ℓ is a constant with units of length, and $|0\rangle$ is the ground state of the oscillator.

1. (5 points) Show that at $t = 0$, the expectation values of the position and momentum operators are $\langle x\rangle = \ell$ and $\langle p\rangle = 0$, respectively.
2. (5 points) Using Ehrenfest theorem, find the expectation values of position and momentum at time t.

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2. (5 points) Find the probability of each energy eigenvalue when the wave function is $\psi(\varphi) = A\cos^2\varphi$, where A is a constant. Recall that $\cos\varphi = (e^{i\varphi} + e^{-i\varphi})/2$.