Problem 1 (30 points)

1st method

A. Determine x'(t') and z'(t'): **4 points**
B. Find time of flight in K': **4 points**
C. Write Lorentz transformations to find x(t') and z(t'): **6 points**
D. Use transformation of time to express t' via t: **6 points**
E. Find x(t) and z(t): **6 points**
F. Determine time of flight in K: **4 points**

2nd method

A. Determine x'(t') and z'(t'): **4 points**
B. Find time of flight in K': **4 points**
C. Find velocity v_x in frame K using Lorentz transformation for velocities: **3 points**
D. Determine the initial v_z(0) in frame K: **3 points**
E. Determine a_z in frame K: **6 points**
F. Find x(t) and z(t): **6 points**
G. Determine time of flight in K: **4 points**

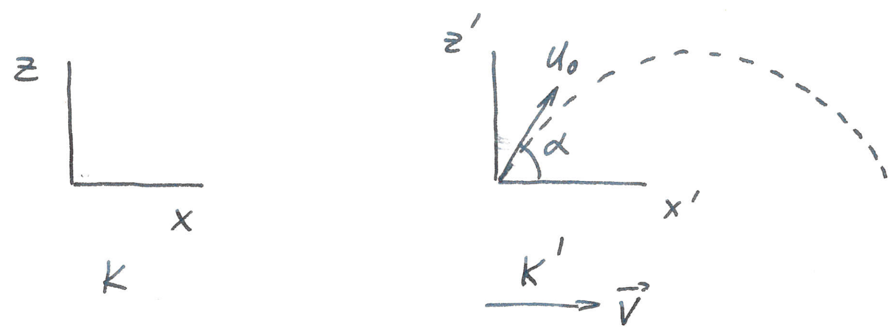
Problem 2 (40 points)

A. Identify m=1 solutions of Laplace's equation in cylindrical coordinates as relevant here: **5 points**
B. Write the form of potential at large distances: **5 points**
C. Identify the character of the boundary condition at r=R: **6 points**
D. Find the constants from the boundary conditions: **5 points**
E. Determine B_r and B_phi from the potential: **5 points**
F. State the amount of force acting on an infinitesimal element of the surface in terms of Maxwell's stress tensor: **5 points**
G. Determine pressure: **5 points**
H. Determine its sign: **4 points**

Problem 3 (30 points)

A. Determine existence of translational invariance and its consequence for the equations (or the induced electric field): **6 points**
B. Write a first (either radial or azimuthal) component of Faraday's law: 7 **points**
C. Integrate the written component of Faraday's law and establish the existence of a constant of integration: **7 points**
D. Write a second (either azimuthal or radial) component of Faraday's law and from the second component determine the value of the integration constant; obtain E_z: **5 points**
E. Explain why other components of the electric field are zero: **5 points**

Problem 1



a) Motion along x' happens with constant velocity:

$$x'(t') = u_0 \cos\alpha\, t'$$

Motion along z' is determined from $\frac{d^2 z'}{dt'^2} = -a_0$,

which gives (in view of $dz'/dt' = u_0 \sin\alpha$ at $t'=0$):

$$z'(t') = u_0 \sin\alpha\, t' - \frac{a_0 t'^2}{2}$$

b) $z' = 0$ at $\boxed{t' = \frac{2u_0 \sin\alpha}{a_0}}$ (and also at $t'=0$)

time of flight in frame K'.

c) Lorentz transformation from K' to K is

$$x = \gamma(x' + Vt') \qquad z = z' \quad , \gamma = \frac{1}{\sqrt{1 - V^2/c^2}}$$

which give

$$x(t') = \gamma(u_0 \cos\alpha + V)t'$$

$$z(t') = u_0 \sin\alpha\, t' - \frac{a_0 t'^2}{2}$$

The obtained expressions, however, give x and z in terms of "old" time t'. To expess t' in terms of t in frame K, we need

$$t = \gamma\left(t' + \frac{V}{c^2}x'\right)$$

or, for our problem

$$t = \gamma t'\left(1 + \frac{V}{c^2}u_0\cos\alpha\right)$$

which gives $t' = \dfrac{t}{\gamma\left(1 + \frac{Vu_0}{c^2}\cos\alpha\right)}$

Substitution into $x(t')$ and $z(t')$ yield the trajectory in frame K:

$$x(t) = \frac{u_0\cos\alpha + V}{1 + \frac{Vu_0}{c^2}\cos\alpha}\,t$$

$$z(t) = u_0\sin\alpha\,\frac{t}{\gamma\left(1 + \frac{Vu_0\cos\alpha}{c^2}\right)} - \frac{a_0 t^2}{2\gamma^2\left(1 + \frac{Vu_0}{c^2}\cos\alpha\right)^2}$$

d) The time of flight is determined from $z=0$:

$$t = \frac{2u_0\sin\alpha}{a_0}\,\gamma\left(1 + \frac{Vu_0}{c^2}\cos\alpha\right)$$

2nd method

Velocity transformation along x:

$$v_x = \frac{v_x' + V}{1 + \frac{v_x' V}{c^2}}$$

with $v_x' = u_0\cos\alpha$ yields the velocity in frame K:

$$v_x = \frac{u_0 \cos\alpha + V}{1 + \frac{u_0 V}{c^2}\cos\alpha}$$

Acceleration (along z) transforms as follows:

$$v_z = \frac{dz}{dt} = \frac{dz'}{\gamma(dt' + \frac{V}{c^2}dx')} = \frac{v_z'}{\gamma(1 + \frac{V v_x'}{c^2})}$$

$$a_z = \frac{dv_z}{dt} = \frac{dv_z}{\gamma(1+\frac{V v_x'}{c^2})} \Big/ \gamma(dt' + \frac{V}{c^2}dx') =$$

$$= \frac{a_z'}{\gamma^2(1+\frac{V v_x'}{c^2})} = -\frac{a_0}{\gamma^2(1+\frac{V u_0}{c^2}\cos\alpha)^2}$$

(where we used that $dv_x' = 0$)

With v_x and a_z determined and $v_z(0)$ given by

$$v_z(0) = \frac{v_z'(0)}{\gamma(1+\frac{V u_0}{c^2}\cos\alpha)},$$ the trajectory is

given by the standard expressions:

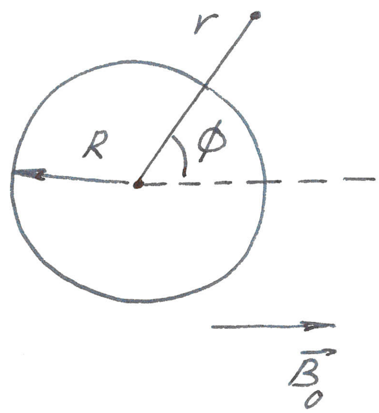
$$x(t) = v_x t = \frac{u_0\cos\alpha + V}{1+\frac{u_0 V}{c^2}\cos\alpha} t$$

$$z(t) = v_z(0)t - \frac{a_z t^2}{2}$$

$$= \frac{u_0 \sin\alpha t}{\gamma(1+\frac{V u_0}{c^2}\cos\alpha)} - \frac{a_0 t^2}{2\gamma^2(1+\frac{V u_0}{c^2}\cos\alpha)^2}$$

Problem 2

a)



In cylindrical coordinates the external field B_0 has the potential

$$\psi_0 = -\vec{B}_0 \cdot \vec{r} = -B_0 r \cos\phi$$

This suggest that the total potential has the form $\psi = f(r)\cos\phi$

From Laplace's equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} = 0$$

we determine that $f(r) = ar + \frac{b}{r}$

Because at $r \to \infty$ the field of the cylinder disappears, $a = -B_0$:

$$\psi(r,\theta) = \left(-B_0 r + \frac{b}{r}\right)\cos\phi$$

The radial component of the magnetic field

$$B_r = -\frac{\partial \psi}{\partial r} = \left(B_0 + \frac{b}{r^2}\right)\cos\phi$$

should vanish everywhere on the surface $r = R$:

$$b = -B_0 R^2$$

so that $\psi(r,\theta) = -B_0\left(r + \frac{R^2}{r}\right)\cos\phi$

The radial component B_r is, therefore,

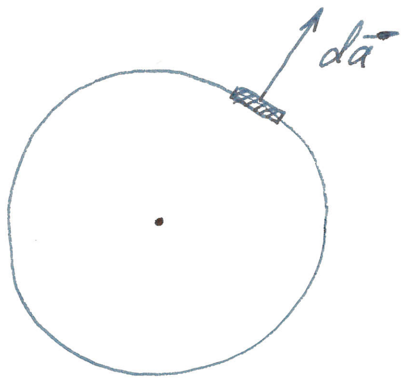
$$B_r = B_0\left(1 - \frac{R^2}{r^2}\right)\cos\phi \quad , \quad r > R$$

The azimuthal component

$$B_\phi = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} = -B_0\left(1 + \frac{R^2}{r^2}\right)\sin\phi \ , \quad r > R$$

Inside the cylinder, $\vec{B} = 0$ $(r < R)$.

b) The force acting on an element $d\vec{a}$ of the surface of the cylinder is given by



$$dF_j = \sigma_{jk}\, da_k$$

where Maxwell's stress tensor

$$\sigma_{jk} = \frac{1}{\mu_0}\left(B_j B_k - \frac{B^2}{2}\delta_{jk}\right)$$

$$dF_j = \frac{1}{\mu_0} B_j \underbrace{B_k da_k} - \frac{1}{2\mu_0} B^2 da_j$$

$\vec{B} \cdot d\vec{a} = 0$ (because $d\vec{a}$ and $\vec{B}$ are orthogonal at the surface of the cylinder)

Thus $\quad d\vec{F} = -\frac{B^2}{2\mu_0} d\vec{a} = -\frac{B_\phi^2}{2\mu_0} d\vec{a}$

The pressure is inward and equal to

$$p = \frac{dF}{da} = -\frac{B_\phi^2}{2\mu_0}\bigg|_{r=R} = -\frac{2B_0^2}{\mu_0}\sin^2\phi$$

Problem 3

Faraday's law is $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Because the system has translational symmetry with respect to the z-axis, the induced field $\vec{E}$ can depend only on r and ϕ: $\vec{E}(r, \phi)$

Consider r-component of Faraday's law

$$\frac{1}{r}\frac{\partial E_z}{\partial \phi} = -\frac{\partial B_r}{\partial t} = -\dot{B}_0\left(1-\frac{R^2}{r^2}\right)\cos\phi$$

where $\dot{B}_0 = \frac{dB_0}{dt}$

Integrating over ϕ, we find $\left(\int \cos\phi \, d\phi = \sin\phi + C \right)$

$$E_z = -\dot{B}_0 r\left(1-\frac{R^2}{r^2}\right)(\sin\phi + C) \qquad (*)$$

The ϕ-component of Faraday's law is

$$-\frac{\partial E_z}{\partial r} = -\frac{\partial B_\phi}{\partial t} = \dot{B}_0\left(1+\frac{R^2}{r^2}\right)\sin\phi \qquad (**)$$

Substituting Eq. (*) into Eq. (**), we conclude that $C = 0$.

Thus

$$\boxed{E_z = -\dot{B}_0\left(r - \frac{R^2}{r}\right)\sin\phi}$$

The other components of $\vec{E}$ are zero:

$$E_r = E_\phi = 0$$

Strictly speaking, this follows from the z-component of Faraday's law

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)-\frac{1}{r}\frac{\partial E_r}{\partial \phi}=0$$

together with $\nabla\cdot\vec{E}=0$ (there are no charge density in the system)

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_r)+\frac{1}{r}\frac{\partial E_\phi}{\partial \phi}=0$$

whose solutions are $E_r=E_\phi=0$ (from the uniqueness of the solutions of Maxwell's equations it follows that there are no other possibilities here).